CLASSIFICATION OF LINKS UP TO SELF #-MOVE

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Dedicated to Professor Shin'ich Suzuki for his 60th birthday

Abstract

A pass-move and a #-move are local moves on oriented links defined by L.H. Kauffman and H. Murakami respectively. Two links are self pass-equivalent (resp. self #-equivalent) if one can be deformed into the other by pass-moves (resp. #-moves), where non of them can occur between distinct components of the link. These relations are equivalence relations on ordered oriented links and stronger than link-homotopy defined by J. Milnor. We give two complete classifications of links with arbitrarily many components up to self pass-equivalence and up to self #-equivalence respectively. So our classifications give subdivisions of link-homotopy classes.

1. Introduction

We shall work in piecewise linear category. All links will be assumed to be ordered and oriented.

A *pass-move* [4] (resp. *#-move* [6]) is a local move on oriented links as illustrated in Figure 1.1(a) (resp. 1.1(b)). If the four strands in Figure 1.1(a) (resp. 1.1(b)) belong to the same component of a link, we call it a *self pass-move* (resp. *self #-move*) [1], [12], [13]. We note that the first author called pass-move and #-move #(II)-move and #(I)-move respectively in his prior papers [12], [13], [14], etc. Two links are *self pass-equivalent* (resp. *self #-equivalent*) if one can be deformed into the other by a finite sequence of self pass-moves (resp. self #-moves). Two links are *link-homotopic* if one can be deformed into the other by finite sequence of *self crossing changes* [5]. Since both self pass-move and self #-move are realized by self crossing changes, self pass-equivalence and self #-equivalence are

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stronger than link-homotopy. Link-homotopy classification is already done by N. Habegger and X.S. Lin [2]. In this paper we give two complete classifications of links with arbitrarily many components up to self pass-equivalence and up to self #-equivalence respectively. So our classifications give subdivisions of link-homotopy classes.

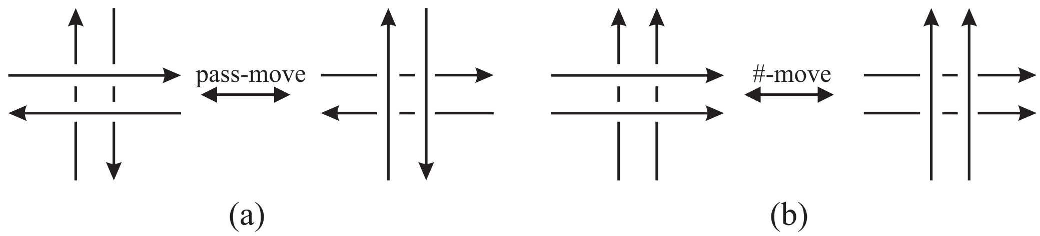

Figure 1.1

An n-component link $l = k_1 \cup \cdots \cup k_n$ is *proper* if the linking number $\mathrm{lk}(l - k_i, k_i)$ is even for any $i(= 1, ..., n)$. We define that a knot is a proper link. For a proper link $l = k_1 \cup \cdots \cup k_n$, we call $\mathrm{Arf}(l) - \sum_{i=1}^{n} \mathrm{Arf}(k_i)$ (mod 2) the *reduced Arf invariant* [12] and denote it by $\overline{\mathrm{Arf}}(l)$, where Arf is the *Arf invariant* [10].

Theorem 1.1. *Let $l = k_1 \cup \cdots \cup k_n$ and $l' = k'_1 \cup \cdots \cup k'_n$ be n-component links. Then the following* (i) *and* (ii) *hold.*

(i) *l and l' are self pass-equivalent if and only if they are link-homotopic and* $\mathrm{Arf}(k_{i_1} \cup \cdots \cup k_{i_p}) = \mathrm{Arf}(k'_{i_1} \cup \cdots \cup k'_{i_p})$ *for any proper links $k_{i_1} \cup \cdots \cup k_{i_p} \subseteq l$ and $k'_{i_1} \cup \cdots \cup k'_{i_p} \subseteq l'$.*

(ii) *l and l' are self #-equivalent if and only if they are link-homotopic and* $\overline{\mathrm{Arf}}(k_{i_1} \cup \cdots \cup k_{i_p}) = \overline{\mathrm{Arf}}(k'_{i_1} \cup \cdots \cup k'_{i_p})$ *for any proper links $k_{i_1} \cup \cdots \cup k_{i_p} \subseteq l$ and $k'_{i_1} \cup \cdots \cup k'_{i_p} \subseteq l'$.*

For two-component links, both self pass-equivalence classification and self #-equivalence classification are done by the first author [13]. His proof can be applied to only two-component links. So we need different approach to proving Theorem 1.1.

A link $l = k_1 \cup \cdots \cup k_n$ is $\mathbb{Z}_2$*-algebraically split* if $\mathrm{lk}(k_i, k_j)$ is even for any i, j $(1 \leq i < j \leq n)$. We note that if $l = k_1 \cup \cdots \cup k_n$ is $\mathbb{Z}_2$-algebraically split link, then l and $k_i \cup k_j$ $(1 \leq i < j \leq n)$ are proper.

Theorem 1.2. *Let $l = k_1 \cup \cdots \cup k_n$ and $l' = k'_1 \cup \cdots \cup k'_n$ be n-component $\mathbb{Z}_2$-algebraically split links. If l and l' are link-homotopic, then*

$$\overline{\mathrm{Arf}}(l) + \sum_{1 \leq i < j \leq n} \overline{\mathrm{Arf}}(k_i \cup k_j) \equiv \overline{\mathrm{Arf}}(l') + \sum_{1 \leq i < j \leq n} \overline{\mathrm{Arf}}(k'_i \cup k'_j) \ (mod\ 2).$$

2. Preliminaries

In this section, we collect several results in order to prove Theorems 1.1 and 1.2.

Let $l = k_1 \cup \cdots \cup k_n$ and $l' = k'_1 \cup \cdots \cup k'_n$ be n-component links. Suppose that there is a disjoint union $\mathcal{A} = A_1 \cup \cdots \cup A_n$ of n annuli in $S^3 \times [0,1]$ with $(\partial(S^3 \times [0,1]), \partial A_i) = (S^3 \times \{0\}, k_i) \cup (-S^3 \times \{1\}, -k'_i)$ $(i = 1, ..., n)$ such that

(i) $\mathcal{A}$ is locally flat except for finite points $p_1, ..., p_m$ in the interior of $\mathcal{A}$, and

(ii) for each p_j $(j = 1, 2, ..., m)$, there is a small neighborhood $N(p_j)$ of p_j in $S^3 \times [0,1]$ such that $(\partial N(p_j), \partial(N(p_j) \cap \mathcal{A}))$ is a link as illustrated in Figure 2.1,

where $-X$ denotes X with the opposite orientation. Then $\mathcal{A}$ is called a *pass-annuli* between l and l'.

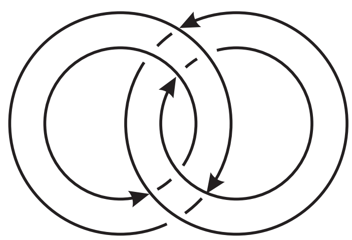

Figure 2.1

The following is proved by the first author in [12].

Lemma 2.1. *Two links l and l' are self pass-equivalent if and only if there is a pass-annuli between them.* $\square$

It is known that a pass-move is realized by a finite sequence of #-moves [8]. Thus we have the following.

Lemma 2.2. *If two links l and l' are self pass-equivalent, then they are self #-equivalent.* $\square$

A Γ-*move* [4] is a local move on oriented links as illustrated in Figure 2.2.

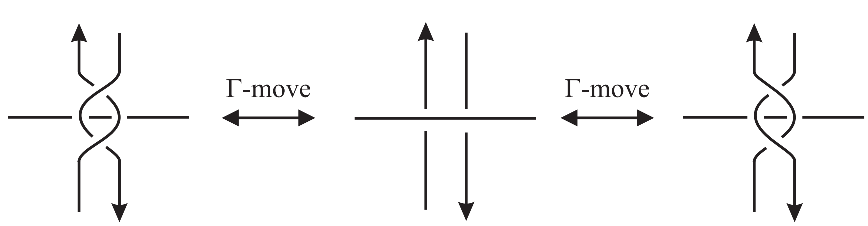

Figure 2.2

The following is known [4].

Lemma 2.3. *A Γ-move is realized by a single pass-move.* $\square$

Let $l = k_1 \cup \cdots \cup k_n$ and $l' = k'_1 \cup \cdots \cup k'_n$ be n-component links such that there is a 3-ball B^3 in S^3 with $B^3 \cap (l \cup l') = l$. Let $b_1, ..., b_n$ be mutually disjoint disks in S^3 such that $b_i \cap l = \partial b_i \cap k_i$ and $b_i \cap l' = \partial b_i \cap k'_i$ are arcs for each i. Then the link $l \cup l' \cup (\bigcup_{i=1}^n \partial b_i) - (\bigcup \mathrm{int}(b_i \cap (l \cup l')))$ is called a *band sum* (or a *product fusion* [11]) of l and l' and denoted by $(k_1 \#_{b_1} k'_1) \cup \cdots \cup (k_n \#_{b_n} k'_n)$. Note that a band sum of l and l' is $\mathbb{Z}_2$-algebraically split if $\mathrm{lk}(k_i, k_j) \equiv \mathrm{lk}(k'_i, k'_j) \pmod 2$ $(1 \le i < j \le n)$.

The following is proved by the first author in [11].

Lemma 2.4. *Two links l and l' are link-homotopic if and only if there is a band sum of l and $-\overline{l'}$ that is link-homotopic to a trivial link, where $(S^3, -\overline{l'}) \cong (-S^3, -l')$.* $\square$

By the definition of the Arf invariant via 4-dimensional topology [10], we have the following.

Lemma 2.5. *Let l and l' be proper links and L a band sum of l and $-\overline{l'}$. Then L is proper and* $\mathrm{Arf}(L) \equiv \mathrm{Arf}(l) + \mathrm{Arf}(l')$ *(mod* 2). $\square$

The following lemma forms an interesting contrast to the lemma above.

Lemma 2.6. *Let $l = k_1 \cup k_2$ and $l' = k'_1 \cup k'_2$ be 2-component links with* $\mathrm{lk}(k_1, k_2)$ *and* $\mathrm{lk}(k'_1, k'_2)$ *odd. Let $L = (k_1 \#_{b_1} (-\overline{k'_1})) \cup (k_2 \#_{b_2} (-\overline{k'_2}))$ be a band sum and L' a band sum obtained from L by adding a single full-twist to b_2; see Figure* 2.3*. Then L and L' are proper and link-homotopic, and* $\mathrm{Arf}(L) \neq \mathrm{Arf}(L')$.

Figure 2.3

Proof. Clearly L and L' are proper and link-homotopic. So we shall show $\mathrm{Arf}(L) \neq \mathrm{Arf}(L')$.

Let a_i be the ith coefficient of the Conway polynomial. Then we have

$$a_3(L) - a_3(L') = a_2((k_1 \#_{b_1} (-\overline{k'_1})) \cup k_2 \cup (-\overline{k'_2})).$$

It is known that the third coefficient of the Conway polynomial of two-component proper link is mod 2 congruent to the sum of the Arf invariants of the link and the components

[7]. This and Lemma 2.5 imply $\mathrm{Arf}(L) - \mathrm{Arf}(L') \equiv a_3(L) - a_3(L') \pmod 2$. By [3],

$$\begin{aligned} & a_2((k_1 \#_{b_1} (-\overline{k_1'})) \cup k_2 \cup (-\overline{k_2'})) \\ & \quad = \mathrm{lk}(k_1 \#_{b_1} (-\overline{k_1'}), k_2)\mathrm{lk}(k_2, -\overline{k_2'}) + \mathrm{lk}(k_2, -\overline{k_2'})\mathrm{lk}(-\overline{k_2'}, k_1 \#_{b_1} (-\overline{k_1'})) \\ & \qquad + \mathrm{lk}(-\overline{k_2'}, k_1 \#_{b_1} (-\overline{k_1'}))\mathrm{lk}(k_1 \#_{b_1} (-\overline{k_1'}), k_2). \end{aligned}$$

Thus we have $\mathrm{Arf}(L) - \mathrm{Arf}(L') \equiv 1 \pmod 2$. □

A Δ-move [8] is a local move on links as illustrated in Figure 2.4. If at least two of the three strands in Figure 2.4 belong to the same component of a link, we call it a *quasi self Δ-move* [9]. Two links are *quasi self Δ-equivalent* if one can be deformed into the other by a finite sequence of quasi self Δ-moves.

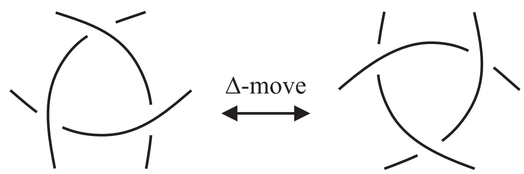

Figure 2.4

The following is proved by Y. Nakanishi and the first author in [9].

Lemma 2.7. *Two links are link-homotopic if and only if they are quasi self Δ-equivalent.* □

3. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.2. Since l is link-homotopic to l', by Lemma 2.7, l is quasi self Δ-equivalent to l'. It is sufficient to consider the case that l' is obtained from l by a single quasi self Δ-move.

Suppose that the three strands of the Δ-move that is applied to the deformation from l into l' belong to one component of l. Without loss of generality we may assume that the component is k_1. Note that k_i and k_i' are ambient isotopic for any $i (\neq 1)$, and that $k_i \cup k_j$ and $k_i' \cup k_j'$ are ambient isotopic for any $i < j$ $(i \neq 1)$. Since a Δ-move changes the value of Arf invariant [8], we have $\mathrm{Arf}(l) \neq \mathrm{Arf}(l')$, $\mathrm{Arf}(k_1) \neq \mathrm{Arf}(k_1')$ and $\mathrm{Arf}(k_1 \cup k_j) \neq \mathrm{Arf}(k_1' \cup k_j')$. Thus we have $\overline{\mathrm{Arf}}(l) = \overline{\mathrm{Arf}}(l')$ and $\overline{\mathrm{Arf}}(k_1 \cup k_j) = \overline{\mathrm{Arf}}(k_1' \cup k_j')$. So we have the conclusion.

We consider the other case, i.e., the three strands of the Δ-move belong to exactly two components of l. Without loss of generality we may assume that the two components

are k_1 and k_2. Note that k_i and k_i' are ambient isotopic for any i, and that $k_i \cup k_j$ and $k_i' \cup k_j'$ are ambient isotopic for any $i < j$ $((i,j) \neq (1,2))$. Since $\mathrm{Arf}(l) \neq \mathrm{Arf}(l')$ and $\mathrm{Arf}(k_1 \cup k_2) \neq \mathrm{Arf}(k_1' \cup k_2')$, $\mathrm{Arf}(l) + \mathrm{Arf}(k_1 \cup k_2) \equiv \mathrm{Arf}(l') + \mathrm{Arf}(k_1' \cup k_2')$ (mod 2). This completes the proof. $\square$

Lemma 3.1. *Let $l = k_1 \cup \cdots \cup k_n$ and $l' = k_1' \cup \cdots \cup k_n'$ be n-component $\mathbb{Z}_2$-algebraically split links. If l and l' are link-homotopic,* $\mathrm{Arf}(k_i) = \mathrm{Arf}(k_i')$ $(i = 1, ..., n)$ *and* $\mathrm{Arf}(k_i \cup k_j) = \mathrm{Arf}(k_i' \cup k_j')$ $(1 \leq i < j \leq n)$*, then l and l' are self pass-equivalent.*

Proof. Since l is link-homotopic to l', by Lemma 2.7, l is quasi self Δ-equivalent to l'. Let u be the minimum number of quasi self Δ-moves which are needed to deform l into l'. By Theorem 1.2, $\mathrm{Arf}(l) = \mathrm{Arf}(l')$. Since a Δ-move changes the value of the Arf invariant, u is even. It is sufficient to consider the case $u = 2$. Therefore there is a union $\mathcal{A} = A_1 \cup \cdots \cup A_n$ of *level-preserving* n annuli in $S^3 \times [0,1]$ with $(\partial(S^3 \times [0,1]), \partial A_i) = (S^3 \times \{0\}, k_i) \cup (-S^3 \times \{1\}, -k_i')$ $(i = 1, ..., n)$ such that

(i) $\mathcal{A}$ is locally flat except for exactly two points p_1, p_2 in the interior of $\mathcal{A}$, and

(ii) for each p_t $(t = 1, 2)$ there is a small neighborhood $N(p_t)$ of p_t in $S^3 \times [0,1]$ such that $(\partial N(p_t), \partial(N(p_t) \cap \mathcal{A}))$ is the Borromean ring R_t, at least two components of which belong to some A_i.

A singular points p_t is called *type* (i) if the three components of R_t belong to A_i and *type* (i,j) $(i < j)$ if one or two componets are in A_i and the others in A_j. For each i (resp. i, j), let u_i (resp. $u_{i,j}$) be the number of the singular points of type (i) (resp. type (i,j)). We note that a number of Δ-moves which are needed to deform k_i into k_i' (resp. $k_i \cup k_j$ into $k_i' \cup k_j'$) is equal to u_i (resp. $u_{i,j} + u_i + u_j$). By the hypothesis of this lemma, we have u_i and $u_{i,j} + u_i + u_j$ are even. Hence u_i and $u_{i,j}$ are even. This implies that both p_1 and p_2 are the same type.

Suppose that p_1 and p_2 are type (i,j). Without loss of generality we may assume that $(i,j) = (1,2)$ and two components of the Borromean ring R_1 belong to A_2. Let α be an arc in the interior of A_1 that connects two singular points p_1 and p_2 of type $(1,2)$, and let $(S^3, L) = (\partial N(\alpha), \partial(N(\alpha) \cap (A_1 \cup A_2)))$. Then L is a 5-component link as illustrated in either Figure 3.1(a) or (b). In the case that L is as Figure 3.1(a), we can deform L into a trivial link by applying Γ-moves to the sublink $L \cap A_2$; see Figure 3.2. In the case that L is as Figure 3.1(b), we can deform L into the link as in Figure 3.2(a) by two Γ-moves, one is applied to $L \cap A_1$ and the other to $L \cap A_2$; see Figure 3.3. It follows from this and Figure

3.2 that L can be deformed into a trivial link by Γ-moves, one is applied to $L \cap A_1$ and the others to $L \cap A_2$.

Suppose that p_1 and p_2 are type (i). Let α be an arc in the interior of A_i that connects two singular points p_1 and p_2 of type (i), and let $(S^3, L) = (\partial N(\alpha), \partial(N(\alpha) \cap A_i))$. By the argument similar to that in the above, L can be deformed into a trivial link by applying Γ-moves to $L \cap A_i$.

Therefore, by Lemma 2.3, we can constract pass-annuli in $S^3 \times [0, 1]$ between l and l'. Lemma 2.1 completes the proof. $\square$

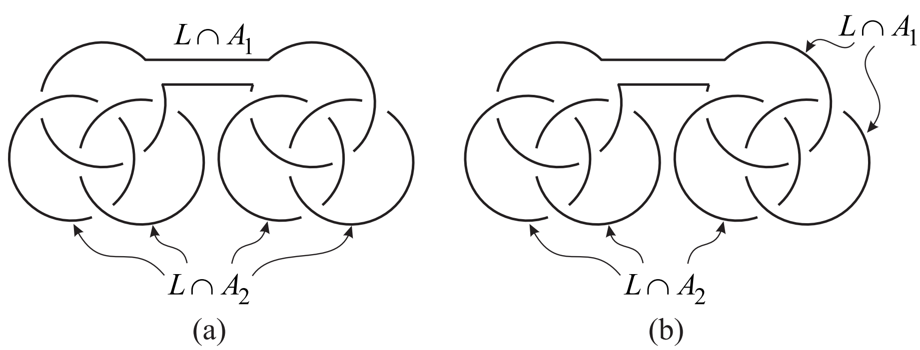

Figure 3.1

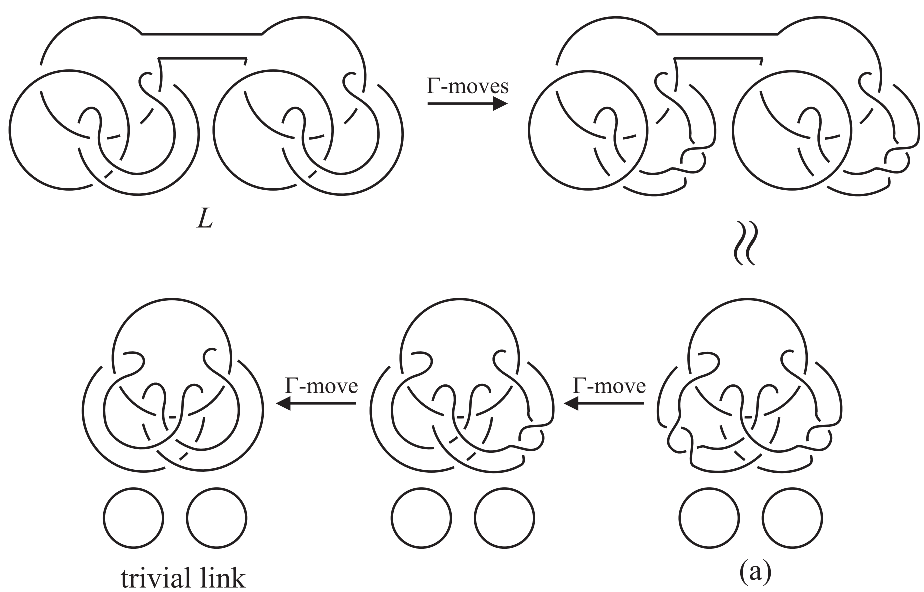

Figure 3.2

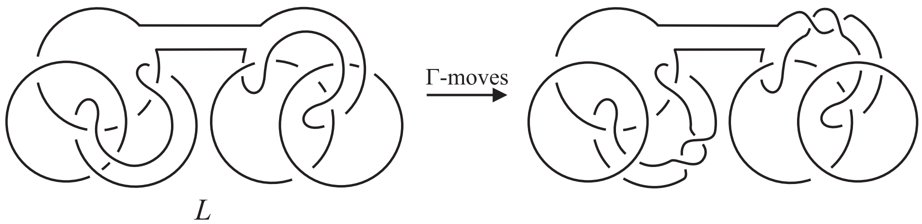

Figure 3.3

Proof of Theorem 1.1. Since both self pass-move and self #-move realized by link-homotopy, the 'only if' parts of (i) and (ii) follow from [13, Proposition]. We shall prove the 'if' parts.

(i) For a link $l = k_1 \cup \cdots k_n$, let G_l^o (resp. G_l^e) be a graph with the vertex set $\{k_1, ..., k_n\}$ and the edge set $\{k_ik_j | \mathrm{lk}(k_i, k_j) \text{ is odd}\}$ (resp. $\{k_ik_j | \mathrm{lk}(k_i, k_j) \text{ is even}\}$). Note that $G_l^o \cup G_l^e$ is the complete graph with n vertices. For a band sum $L = K_1 \cup \cdots \cup K_n (= (k_1 \#_{b_1} (-\overline{k'}_1)) \cup \cdots \cup (k_n \#_{b_n} (-\overline{k'}_n)))$ of l and $-\overline{l'}$, let A_L be a graph with the vertex set $\{K_1, ..., K_n\}$ and the edge set $\{K_iK_j | \mathrm{Arf}(K_i \cup K_j) = 0\}$. (Note that L is a $\mathbb{Z}_2$-algebraically split link since l and l' are link-homotopic.)

Claim. There is a band sum L of l and $-\overline{l'}$ such that L is link-homotopic to a trivial link and A_L is the complete graph with n vertices.

Proof. Let T be a maximal subgraph of G_l^o that does not contain a cycle. Since T does not contain a cycle, by Lemmas 2.4 and 2.6, there is a band sum L of l and l' such that L is link-homotopic to a trivial link and $T \subset h(A_L)$, where $h : A_L \longrightarrow G_l^o \cup G_l^e$ the natural map defined by $h(K_i) = k_i$ and $h(K_iK_j) = k_ik_j$. By Lemma 2.5, we have $G_l^e \subset h(A_L)$. Since h is injective and $G_l^o \cup G_l^e$ is the complete graph, it is sufficient to prove that h is surjective. Let E be the set of edges which are not contained in $h(A_L)$, and $H^o = h(A_L) \cap G_l^o$. Suppose $E \neq \emptyset$. Then there is an edge $e \in E$ such that there is a cycle C in $H^o \cup e$ containning e whose any 'diagonals' are not contained in G_l^o. (In fact, for each $e_i \in E$, consider the minimum length l_i of cycles in $H^o \cup e_i$ containning e_i and choose an edge e and a cycle C in $H^o \cup e$ containning e so that length C is equal to $\min\{l_i | e_i \in E\}$.) Without loss of generality we may assume that $C = k_1k_2...k_ck_1$ and $e = k_1k_2$. Set $l_c = k_1 \cup \cdots \cup k_c$ and $L_c = K_1 \cup \cdots \cup K_c$. Since C has no diagonals in G_l^o, all diagonals are in G_l^e. Thus we have $k_ik_j \subset H^o \cup G_l^e (= h(A_L))$ for any i, j $(1 \leq i < j \leq c)$ except for $(i, j) = (1, 2)$. This implies $\mathrm{Arf}(K_i \cup K_j) = 0$ for any i, j $(1 \leq i < j \leq c,\ (i, j) \neq (1, 2))$. The fact that C has no diagonals in G_l^o implies l_c is a propre link. By the hypothesis about the Arf invariants and Lemma 2.5, we have $\mathrm{Arf}(L_c) \equiv 2\mathrm{Arf}(l_c) \equiv 0 \pmod 2$ and $\mathrm{Arf}(K_i) \equiv 2\mathrm{Arf}(k_i) \equiv 0 \pmod 2$

$(i = 1, ..., c)$. Since L_c is link-homotopic to a trivial link, by Theorem 1.2, $\mathrm{Arf}(K_1 \cup K_2) = 0$. This contradicts $e = k_1k_2 \in E$. $\square$

By Claim, there is a band sum $L = K_1 \cup \cdots \cup K_n$ of l and $-\overline{l'}$ such that L is link-homotopic to a trivial link, $\mathrm{Arf}(K_i) = 0$ $(i = 1, ..., n)$ and $\mathrm{Arf}(K_i \cup K_j) = 0$ $(1 \leq i < j \leq n)$. By Lemma 3.1, L is self pass-equivalent to a trivial link. Since L is a band sum of l and $-\overline{l'}$, we can constract a pass-annuli between l and l'. Lemma 2.1 completes the proof.

(ii) Since a #-move changes the value of the Arf invariant [6], by applying self #-moves, we may assume that $\mathrm{Arf}(k_i) = \mathrm{Arf}(k_i')$ for any i. Theorem 1.1(i) and Lemma 2.2 complete the proof. $\square$

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